Applied Mathematical Sciences, Vol. 8, 2014, no. 164, 8153 - 8170
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ams.2014.46408

On Some New Families Based on the Exponential Better than Used Concept

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Abstract

New families of life distributions Based on the Exponential Better than Used Concept are introduced. Definitions and Basic Results are stated. Several properties of these classes are presented, including the preservation under convolution and mixing. Testing exponentially against exponential better (worse) than used in convex order (2) ,EBUC(2), is proposed by using moment inequalities. The Pitman asymptotic efficiency (PAE) is discussed. Selected critical values are tabulated for selected sample sizes. Application with real data is given to illustrate the theoretical results.

Keywords: Exponential, convolution, mixing and EBU

1 Introduction

The applications of classes of life distributions can be seen in reliability, engineering, replacement analysis, inventory and queuing theory, biological science, maintenance and biometrics. Various classes of life distributions and their dual have been introduced in reliability to describe several types of deterioration (improvements). Therefore, statisticians and reliability analysts have found it useful to categorize life distributions according to different aging properties.

These classes have been considered by different authors and based on two concepts:

(i) The failure rate classes, e.g. IFR, IFRA, NBUFR,…,etc.
(ii) The conditional survival classes, e.g., NBU, NBUE, HNBUE, NBUC, NBUCA, DMRL,…,etc.

Elbatal (2002) introduced a class of life distribution called, exponential better than used (EBU) and its dual class (EWU). He investigated their relationship to other classes of life distribution. He discussed the closure properties under reliability operations, moment inequality and under shock model. Mahmoud et al. (2005) introduced a class of life distribution called, exponential better than used average EBUA and its dual EWUA. He discussed the closure properties under reliability operations, formation of parallel system and under shock model.

2 Definitions and Basic Results

Some classes of life distributions are now introduced. Let X and Y be two non-negative random variables with distribution functions F(x) and F(Y), and survival functions$\bar{F}(X)$ and$\bar{F}(Y)$, respectively. Define$N = \{0,1, \ldots\}$, let $\{P_k, K\epsilon N\}$ be a discrete distribution and define the probability of survival as $\bar{P}_k = 1 - P_k$ with $\bar{P}_0 = 1$ and $P_k = \bar{P}_k - \bar{P}_{k-1}$, where $k \geq 1$. Clearly, $\bar{P}_k \geq \bar{P}_{k+1} \geq \bar{P}_{k+2} \geq \cdots$

Definition 2.1
A discrete distribution *P* with finite mean μ is called geometric better than used or GBU (geometric worse than used or GWU) if

$$\bar{P}(i+j) \leq (\geq) \bar{P}(j) \left(1 - \frac{1}{\mu}\right)^i \ \forall\, i, j \in N \,. \tag{1}$$

Definition 2.2
A life distribution F (i.e., F (0) =0) is called exponential better than used (2) (EBU2) and its dual EWU2 if

$$\int_0^u \bar{F}(x+t)\, dt \leq (\geq)\ e^{-\frac{x}{\mu}} \int_0^u \bar{F}(t)\, dt \qquad \forall\, u > 0 \,. \tag{2}$$

Definition 2.3
A discrete distribution *P* with finite mean μ is called geometric better than used (2) or GBU2 (geometric worse than used (2) or GWU2) if

$$\sum_{j=0}^{k} \bar{P}(i+j) \leq (\geq) (1 - \frac{1}{\mu})^i \sum_{j=0}^{k} \bar{P}(j) \ \forall\ i, j, \in N. \tag{3}$$

Definition 2.4
A life distributions F i.e., F (0) =0 is called exponential better than used in convex order one EBUC(1) if

$$\int_{u}^{\infty} \bar{F}(x+t)dx \le \mu\, e^{-\frac{u}{\mu}} \bar{F}(t). \quad (4)$$

Definition 2.5
A discrete distribution $\{P_k, k \in \mathrm{N},\}$ is said to be geometric better than used in convex order one GBUC (1) if,

$$\sum_{i=k}^{\infty} \bar{P}(i+j) \le \bar{P}(j) \sum_{i=k}^{\infty} \left(1-\frac{1}{\mu}\right)^{i}. \quad (5)$$

Definition 2.6
A life distribution F i.e. F (0) =0 is called exponential better than used in convex order two EBUC(2) if

$$\int_{u}^{\infty} \bar{F}(x+t)dt \le e^{\frac{-x}{\mu}} \int_{u}^{\infty} \bar{F}(t)\, dt. \quad (6)$$

Definition 2.7
A discrete distribution $\{\bar{P}_k, k \in \mathrm{N},\}$ is said to be geometric better than used in convex order two GBUC (2) if,

$$\sum_{j=k}^{\infty} \bar{P}(i+j) \le \left(1-\frac{1}{\mu}\right)^{i} \sum_{j=k}^{\infty} \bar{P}(j). \quad (7)$$

Definition 2.8
A life distributions F (i.e., F (0) =0) is called exponential better than used in average (2) (EBUA(2)) and its dual EWUA(2) if

$$\int_{0}^{\infty} \int_{0}^{u} \bar{F}(x+t)dx\, dt \le (\ge)\, \mu \int_{0}^{u} e^{-\frac{x}{\mu}} dx. \quad (8)$$

Definition 2.9
A discrete distribution $\{P_k, k \in \mathrm{N},\}$ is said to be GBUA2 (GWUA2) if

$$\sum_{j=0}^{\infty}\sum_{i=0}^{m}\bar{P}(i+j) \le (\ge)\mu \sum_{i=0}^{m}\left(1-\frac{1}{\mu}\right)^{i} . \tag{9}$$

Theorem 2.1
If X is GBU (GWU) then X is D-NBUE (D-NWUE)

Proof. For the GBU case similar arguments hold for the GWU case. If X is GBU then, the summation with respect to *i* over (0, ∞) implies that

$$\sum_{i=0}^{\infty}\bar{P}(i+j) \le (\ge)\bar{P}(j)\sum_{i=0}^{\infty}\left(1-\frac{1}{\mu}\right)^{i} = \mu\bar{P}(j)$$

$$\Leftrightarrow \frac{\sum_{i=0}^{\infty}\bar{P}(i+j)}{\bar{P}(j)} \le (\ge)\mu$$

$$\Leftrightarrow \frac{\sum_{m=j}^{\infty}\bar{P}(m)}{\bar{P}(j)} \le (\ge)\mu$$

Then X is D-NBUE (D-NWUE).
Theorem 2.2
If X is GBU (GWU) then X is D-HNBUE (D-HNWUE)

Proof. X is GBU means that

$$\bar{P}(i+j) \le \bar{P}(j)\left(1-\frac{1}{\mu}\right)^{i} \ \forall\ i,j \in N.$$

By getting summation with respect to *j* over (0, ∞), we get

$$\sum_{j=0}^{\infty}\bar{P}(i+j) \le \left(1-\frac{1}{\mu}\right)^{i}\sum_{j=0}^{\infty}\bar{P}(j) = \mu\left(1-\frac{1}{\mu}\right)^{i}$$

$$\Leftrightarrow \sum_{\boldsymbol{m=i}}^{\infty}\bar{P}(m) \le \mu\left(1-\frac{1}{\mu}\right)^{i} \ \forall\ i \in N,$$

hence X is D-HNBUE.

3 The Closure Properties

3.1 Convolution

As an important reliability operation, convolutions of life distributions of a certain class are often paid much attention. The closure properties of IFR, NBU, NBUE, and IFRA can be found in Barlow and Proschan (1981). The closure properties of class NBUC were pointed out in Cao and Wang (1991) and Hu and Xie (2002). In the next theorem the closure property of the class under the convolution operation is established.

Theorem 3.1
Suppose that P_1 and P_2 are two independent GBU life distributions. Then their convolution is also GBU

Proof.

$$\bar{P}(i+j) = \sum_{z=0}^{\infty} \bar{P}_1\ (i+j-z)p_2(z)$$

$$\bar{P}(i+j) = \sum_{z=0}^{\infty} \bar{P}_1\ (j-z)p_2(z)\left(1-\frac{1}{\mu}\right)^i = \left(1-\frac{1}{\mu}\right)^i \sum_{z=0}^{\infty} \bar{P}_1\ (j-z)p_2(z)$$

$$= \bar{P}(j)\left(1-\frac{1}{\mu}\right)^i.$$

Remark: A similar result can be written for the GWU class.

Theorem 3.2
Suppose that F_1 and F_2 are two independent EBU (2) life distributions. Then their convolution is also EBU (2).
Proof

$$\bar{F}(x+t) = \int_0^{\infty} \overline{F_1}\ (x+t-z)dF_2(z).$$

By integrating both sides with respect to t, then

$$\int_0^u \bar{F}(x+t)dt = \int_0^u \int_0^{\infty} \overline{F_1}(x+t-z)\ dF_2(z)dt$$

$$= \int_0^{\infty} \int_0^u \overline{F_1}(x+t-z)dtd\overline{F_2}(z)$$

$$\leq \int_0^{\infty} e^{-\frac{x}{\mu}} \left[\int_0^u \overline{F_1}(t-z)dt\right] d\overline{F_2}(z)$$

$$\leq e^{-\frac{x}{\mu}} \int_0^u \left[\int_0^{\infty} \overline{F_1}(t-z)dt\right] d\overline{F_2}(z) \leq\ e^{-\frac{x}{\mu}} \int_0^u \bar{F}(t)\ dt.$$

Theorem 3.3
Suppose that P_1 and P_2 are two independent GBU (2) life distributions. Then their convolution is also GBU (2).

Proof.

$$\bar{P}(i+j)=\sum_{z=0}^{\infty}\bar{P}_1\ (i+j-z)p_2(z)$$

By summing with respect to *j*, then

$$\sum_{j=0}^{u}\bar{P}(i+j)=\sum_{j=0}^{u}\sum_{z=0}^{\infty}\bar{P}_1\ (i+j-z)p_2(z)$$
$$=\sum_{z=0}^{\infty}p_2(z)\sum_{j=0}^{u}\bar{P}_1\ (i+j-z)$$
$$\leq\sum_{z=0}^{\infty}p_2(z)\sum_{j=0}^{u}\left(1-\frac{1}{\mu}\right)^i\bar{P}_1\ (j-z)$$
$$=\left(1-\frac{1}{\mu}\right)^i\sum_{j=0}^{u}\sum_{z=0}^{\infty}P_2(z)\,\bar{P}_1(j-z)$$
$$=\left(1-\tfrac{1}{\mu}\right)^i\textstyle\sum_{j=0}^{u}\bar{P}(j).$$

Theorem 3.4
Suppose that F_1 and F_2 are two independent EBUA (2) life distributions. Then their convolution is also EBUA (2).

Proof.

$$\bar{F}(x+t)=\int_0^{\infty}\bar{F}_1(x+t-z)d\,\bar{F}_2(z)$$

By integrating both sides with respect to x and t, then

$$\int_0^{\infty}\int_0^{u}\bar{F}(x+t)dxdt=\int_0^{\infty}\int_0^{u}\int_0^{\infty}\overline{F_1}(x+t-z)\,dF_2dxdt$$
$$\leq\textstyle\int_0^{u}e^{-\frac{x}{\mu}}\int_0^{\infty}\int_0^{\infty}\overline{F_1}(t-z)\,dF_2dtdx$$
$$\leq\int_0^{u}e^{-\frac{x}{\mu}}dx\left[\int_0^{\infty}\bar{F}(t)dt\right]\leq\ \mu\int_0^{u}e^{-\frac{x}{\mu}}dx.$$

This proves that F is also EBUA (2).
Remark: A similar result can be written for the EWUA (2) class.

Theorem 3.5
Suppose that P_1 and P_2 are two independent GBUA2 life distributions. Then their convolution is also GBUA2
Proof.

$$\bar{P}(i+j)=\sum_{z=0}^{\infty}\bar{P}_1\ (i+j-z)p_2(z)$$

$$\sum_{j=0}^{\infty}\sum_{i=0}^{m}\bar{P}(i+j)=\sum_{j=0}^{\infty}\ \sum_{i=0}^{m}\sum_{z=0}^{\infty}\bar{P}_1\ (i+j-z)p_2(z)$$

$$=\sum_{j=0}^{\infty}\sum_{i=0}^{m}\left(1-\frac{1}{\mu}\right)^{i}\sum_{z=0}^{\infty}\bar{P}_1\ (j-z)p_2(z)=\sum_{j=0}^{\infty}\bar{P}(j)\sum_{i=0}^{m}\left(1-\frac{1}{\mu}\right)^{i}=\mu\sum_{i=0}^{m}\left(1-\frac{1}{\mu}\right)^{i}.$$

3.2 Mixtures
Suppose we are given a one-parameter family of life distributions
$\{F_{\propto}(t), t\geq 0\}$, where the parameter $\propto\geq o$. Let $\propto$ be a random variable and let G be its distribution function; then, G is said to be a mixing distribution. The mixture φ of the family $\{F_{\propto}\}$, with respect to G, and the mixture $\bar{\varphi}$ of $\{\bar{F}_{\propto}\}$, are defined by
$\varphi=\int_0^{\infty}F_{\propto}(t)\ dG(\propto), t>0$ and $\bar{\varphi}=\int_0^{\infty}\bar{F}_{\propto}(t)\ dG(\propto), t>0$
As usual, we assume that the family $\{\bar{F}_{\propto}\}$ and the mixing distribution G satisfy some conditions, and we want to drive the resulting mixture φ properties.

Theorem 3.6
Let P be a mixture of $P_l, l\in N$, with each P_l GWU and no two distributions P_l, P_k cross on N. then P is GWU.

Proof. Let $k, l\in N,$ and g be the mixing p.m.f. By Chebyshev's inequality

$$\bar{P}(i)\bar{P}(j)=\left(1-\frac{1}{\mu}\right)^{i}\bar{P}(j)$$

$$=\left[\sum_{k=0}^{\infty}\left(1-\frac{1}{\mu_k}\right)^{i}g(k)\right]\left[\sum_{l=0}^{\infty}\bar{P_l}(j)g(l)\right]$$

$$\leq\sum_{k=0}^{\infty}\left(1-\frac{1}{\mu_k}\right)^{i}\bar{P}_k(j)g(k)\leq\sum_{k=0}^{\infty}\bar{P}_k(i+j)g(k)$$

$$\leq\sum_{k=0}^{\infty}\bar{P}_k(i+j)g(k)=\bar{P}(i+j),$$

which implies that P is GWU.

Theorem 3.7

Let F be a mixture of $F_\propto, \propto \geq o$, with each $F_\propto$ EWU2, and $F_\propto$ no crossing property on (0,∞), then F is EWU2.

Proof. By the Chebyshev's inequality for similarly ordered functions (Hardy, et al, 1952, theorem4.3)

$$e^{\frac{-x}{\mu}}\int_0^u \bar{F}(t)\ dt = \int_o^\infty e^{\frac{-x}{\mu_\propto}}\ dG(\propto)\int_0^u\int_0^\infty \bar{F}_\propto(t)\ dG(\propto)\ dt$$

$$\leq \int_0^u\int_0^\infty e^{\frac{-x}{\mu_\propto}}\ \bar{F}_\propto(t)\ dG(\propto)dt$$

$$\leq \int_0^u\int_0^\infty \bar{F}_\propto(x+t)\ dG(\propto)dt$$

$$\equiv \int_0^u \bar{F}(x+t)\,dt,$$

Then F is EWU2.

Theorem 3.8

Let P be a mixture of $P_l, l \in N$, with each P_l GWU2 and no two distributionsP_l, P_k cross on N. then P is GWU2.

Proof. Let $k, l \in N$, and g be the mixing p. m. f. By Chebyshev's inequality

$$\left[\left(1-\frac{1}{\mu}\right)^i\right]\sum_{j=0}^u \bar{P}(j) = \left[\sum_{k=0}^\infty\left(1-\frac{1}{\mu_k}\right)^i g(k)\right]\left[\sum_{j=0}^u\sum_{l=0}^\infty \bar{P}_l(j)g(l)\right]$$

$$=\left\{\sum_{k=0}^\infty\left[\left(1-\frac{1}{\mu_k}\right)^i g(k)\right]\right\}\left[\sum_{j=0}^u\sum_{l=0}^\infty \bar{P}_l(j)g(l)\right]$$

$$\leq \sum_{j=0}^u\sum_{l=0}^\infty\left[\bar{P}_l(j)\left(1-\frac{1}{\mu_l}\right)^i\right]g(l) = \sum_{j=0}^u \bar{P}(i+j),$$

which implies that F is GWU2.

Theorem 3.9

Let F be a mixture of $F_\propto, \propto \geq o$, with each $F_\propto$ EWUC1, and $F_\propto$ no crossing property on (0,∞), then F is EWUC1.

Proof. By the Chebyshev's inequality for similarly ordered functions (Hardy, et al., 1952, Theorem 4.3)

$$\bar{F}(t)\int_{u}^{\infty} e^{\frac{-x}{\mu}}\,dx = \int_{o}^{\infty} \bar{F}_{\propto}(t)\ dG(\propto)\int_{u}^{\infty}\int_{0}^{\infty} e^{\frac{-x}{\mu_{\propto}}}\,dG(\propto)\,dx$$

$$\leq \int_{u}^{\infty}\int_{0}^{\infty} e^{\frac{-x}{\mu_{\propto}}}\ \bar{F}_{\propto}(t)\ dG(\propto)dx$$

$$\leq \int_{u}^{\infty}\int_{0}^{\infty} \bar{F}_{\propto}(x+t)\ dG(\propto)dx$$

$$\equiv \int_{u}^{\infty} \bar{F}(x+t)\,dx.$$

Then F is EWUC1.

Theorem 3.10

Let P be a mixture of $P_l\,, l \in N,$ with each P_l GWUC1 and no two distributions$P_l\,, P_k$ cross on N. then P is GWUC1.

Proof. Let $k, l \in N,$ and g be the mixing p.m.f. By Cebyshev's inequality

$$\bar{P}(j)\left[\sum_{i=m}^{\infty}\left(1-\frac{1}{\mu}\right)^{i}\right] = \left[\sum_{k=0}^{\infty}\sum_{i=m}^{\infty}\left(1-\frac{1}{\mu_k}\right)^{i} g(k)\right]\left[\sum_{l=0}^{\infty}\bar{P}_l(j)g(l)\right]$$

$$= \left\{\sum_{k=0}^{\infty}\left[\sum_{i=m}^{\infty}\left(1-\frac{1}{\mu_k}\right)^{i} g(k)\right]\right\}\left[\sum_{l=0}^{\infty}\bar{P}_l(j)g(l)\right]$$

$$\leq \sum_{k=0}^{\infty}\left[\overline{P_k}(j)\sum_{i=m}^{\infty}\left(1-\frac{1}{\mu_k}\right)^{i}\right]g(k)$$

$$\leq \sum_{i=m}^{\infty}\sum_{k=0}^{\infty}\left[\overline{P_k}(j)\left(1-\frac{1}{\mu_k}\right)^{i}\right]g(k)$$

$$= \sum_{i=m}^{\infty}\bar{P}(i+j),$$

which implies that F is GWUC1.

Theorem 3.11

Suppose F is the mixture of $F_{\propto}\,, \propto \geq o$, with each $F_{\propto}$ EWUC2, and $F_{\propto}$ no crossing property on (0,∞), then F is EWUC2.

Proof. By the Chebyshev's inequality for similarly ordered functions (Hardy, et al., 1952, Theorem 4.3)

$$e^{\frac{-x}{\mu}} \int_u^\infty \bar{F}(t)\ dt = \int_o^\infty e^{\frac{-x}{\mu_\propto}}\ dG(\propto) \int_u^\infty \int_0^\infty \bar{F}_\propto(t)\ dG(\propto)\ dt$$
$$\le \int_u^\infty \int_0^\infty e^{\frac{-x}{\mu_\propto}}\ \bar{F}_\propto(t)\ dG(\propto) dt$$
$$\le \int_u^\infty \int_0^\infty \bar{F}_\propto(x+t)\ dG(\propto) dt$$
$$\equiv \int_u^\infty \bar{F}(x+t)\, dt,$$

then F is EWU2.

Theorem 3.12

Let P be a mixture of $P_l, l \in N,$ with each P_l GWUC2 and no two distributionsP_l, P_k cross on N. then P is GWUC2.

Proof. Let $k, l \in N,$ and g be the mixing p.m.f. By Chebyshev's inequality

$$\sum_{j=m}^\infty \bar{P}(j) \left[\left(1-\frac{1}{\mu}\right)^i\right] = \left[\sum_{k=0}^\infty \left(1-\frac{1}{\mu_k}\right)^i g(k)\right] \left[\sum_{j=m}^\infty \sum_{l=0}^\infty \bar{P}_l(j) g(l)\right]$$
$$= \left\{\sum_{k=0}^\infty \left[\left(1-\frac{1}{\mu_k}\right)^i g(k)\right]\right\} \left[\sum_{j=m}^\infty \sum_{l=0}^\infty \bar{P}_l(j) g(l)\right]$$
$$\le \sum_{j=m}^\infty \sum_{l=0}^\infty \left[\bar{P}_l(j) \left(1-\frac{1}{\mu_l}\right)^i\right] g(l)$$
$$\le \sum_{j=m}^\infty \sum_{l=0}^\infty \left[\bar{P}_l(j) \left(1-\frac{1}{\mu_l}\right)^i\right] g(l)$$
$$= \sum_{j=m}^\infty \bar{P}(i+j),$$

which implies that F is GWUC2.

Theorem 3.13

Suppose F is the mixture of $F_\propto, \propto \ge o$, with each $F_\propto$ EWUC2, and $F_\propto$ no crossing property on (0,∞), then F is EWUC2.

Proof. By the Chebyshev's inequality for similarly ordered functions (Hardy, et al., 1952, Theorem 4.3)

$$\mu\int_0^u e^{\frac{-x}{\mu}}\,dx = \int_o^\infty \mu_\alpha\ dG(\propto)\int_0^u\int_0^\infty e^{\frac{-x}{\mu_\propto}}\ dG(\propto)\,dx$$
$$\leq \int_0^u\int_0^\infty \mu_\alpha\ e^{\frac{-x}{\mu_\propto}}\,dG(\propto)dx$$
$$\leq \int_0^\infty\int_0^u\int_0^\infty\ \bar{F}_\propto(x+t)\ dG(\propto)dxdt$$
$$\equiv \int_0^u\int_0^\infty \bar{F}(x+t)\,dx\,dt,$$

Then F is EWUA2.

Theorem 3.14

Let P be a mixture of $P_l, l \in N,$ with each P_l GWUA2 and no two distributions P_l, P_k cross on N. then P is GWUA2.

Proof. Let $k, l \in N,$ and g be the mixing p.m.f. By Cebyshev's inequality

$$\mu\left[\sum_{i=0}^m\left(1-\frac{1}{\mu}\right)^i\right] = \left[\sum_{k=0}^\infty\sum_{i=0}^m\left(1-\frac{1}{\mu_k}\right)^i g(k)\right]\left[\sum_{l=0}^\infty \mu_l g(l)\right]$$
$$= \left\{\sum_{k=0}^\infty\left[\sum_{i=0}^m\left(1-\frac{1}{\mu_k}\right)^i g(k)\right]\right\}\left[\sum_{l=0}^\infty \mu_l g(l)\right]$$
$$\leq \sum_{k=0}^\infty\left[\sum_{i=0}^m \mu_k\left(1-\frac{1}{\mu_k}\right)^i\right]g(k)$$
$$\leq \sum_{k=0}^\infty\left[\sum_{i=0}^m\sum_{j=0}^\infty \overline{P_k}(j)\left(1-\frac{1}{\mu_k}\right)^i\right]g(k)$$
$$\leq \sum_{j=0}^\infty\sum_{i=0}^m\sum_{k=0}^\infty\left[\overline{P_k}(j)\left(1-\frac{1}{\mu_k}\right)^i\right]g(k)$$
$$= \sum_{j=0}^\infty\sum_{i=0}^m \bar{P}(i+j),$$

which implies that F is GWUA2.

4 Testing Exponentiality AgainstEBUC(2)

In the context of reliability and life testing, the failure rate of a life distribution plays an important role for stochastic modeling and classification. Being a ratio of probability density function and the corresponding survival function, it uniquely determines the underlying distribution and exhibits different monotonic Behaviors. The concept of the ageless notion is equivalent to the phenomenon that age has no effect on the failure rate. Thus, the ageless property is equal to constant failure rate, that is, the distribution is exponential. Hence, testing non-parametric classes is done by testing exponentiality versus some kind of classes. This applies to many non-parametric classes such as the NBU, NBUC, NBUCA and NBUE, among others.

Moments inequalities for some classes of life distributions have been recently appeared and hence have been used in testing. Ahmad (2001) used the new testing against IFR, NBU and NBUE. Testing IFRA, NBUC and DMRL based on moments inequalities have been studied by Ahmad and Mugdadi (2002). Abdul-Moniem (2007), discussed the test against EBUCA. Here, we will use the same methods that have been addressed for the class of EBUC(2), and apply it for testing.

4.1 Moments Inequalities for EBUC(2)

In the following, exponentiality against EBUC(2) based on the moment inequalities is tested.

Theorem 4.1
Suppose that F isEBUC(2) life distribution such that itsμ_{r+s+4}, the moment of order, is finite (r+s+4) for some integers r and s , then the following moment inequality holds

$$\frac{\mu_{(r+s+4)}}{(r+s+4)!} \leq (\geq) \frac{\mu^{r+1}\mu_{s+2}}{(s+2)!}. \tag{10}$$

Proof. We have, for any integer $s \geq 0$:

$$\int_0^\infty x^s\,\bar{F}(x)dx = E\left[\int_0^x x^s\,dx\right] = \frac{1}{s+1}E(x^{s+1}) = \frac{\mu_{(s+1)}}{s+1}.$$

Multiplying both sides of (6) by $x^r u^s$and integrating we get, R.H.S

$$\int_0^\infty \int_0^\infty x^r u^s e^{\frac{-x}{\mu}} \int_u^\infty \bar{F}(t)\,dtdudx$$

$$= \int_0^\infty x^r e^{\frac{-x}{\mu}} \int_0^\infty u^s \int_u^\infty \bar{F}(t)\, dtdudx$$
$$= r!\ \mu^{r+1} \int_0^\infty \bar{F}(u) \int_0^u t^s dtdu$$
$$= \frac{r!\ \mu^{r+1}}{s+1} \int_0^\infty u^{s+1}\bar{F}(t)\, du$$
$$= \frac{r!\, \mu^{r+1} \mu_{s+2}}{(s+1)(s+2)}.$$

Using methods similar to those used in proving "Theorem 10" of Mugdadi and Ahmad (2005), it can be shown that the left-hand side is equal to

$$\frac{r!\, s!\, \mu_{(r+s+4)}}{(r+s+4)!}.$$

Now by using inequality (10), can be tested the null hypothesis
H_0: F is exponential versus,
H_1: F is EBUC(2) and not exponential.
According to inequality (10), we set the following measure of departure from *Ho:*

$$\delta(r,s) = \frac{\mu^{r+1} \mu_{s+2}}{(s+2)!} - \frac{\mu_{(r+s+4)}}{(r+s+4)!}. \tag{11}$$

The choice of r and s is a question that needs to be addressed. There are two possible routes. Either to choose a small values like r = 0, s = 0 or s = 1, r = 1 to make calculations simple or to try to find the values of r and s that give the maximum power or efficiency if we have some belief about an alternative. To choose r and s that maximize the power one can use empirical calculations by simulating sampling from an alternative distribution calculating the empirical powers for various sample sizes at r = 0, 1, … and s = 0, 1, … etc. and choose the values of r and s that give the best power, see, Shokry (2009).
Putting r = s =0 in (11), the following may be used as a measure of departure from H_0 in favor of H_1

$$\delta_E = 12\mu\ \mu_2 - \mu_4 \geq (\leq) 0. \tag{12}$$

Note that under $H_0: \delta_E = 0$, while under $H_1: \delta_E > (<)0$. Thus to estimate δ_E by $\hat{\delta}_E$. Let $X_1, \ldots, X_n$ be a random sample from F and μ is estimated by $\bar{X}$, where $\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$ is the usual sample mean. Then, $\hat{\delta}_E$ is given by using (12) as

$$\hat{\delta}_E = \frac{1}{n^2}\sum_i \sum_j \left\{12\, X_i X_j^2 - X_i^4\right\}$$
$$= n^{-2}\sum_i \sum_j \emptyset\left(X_i, X_j\right),$$

where

$$\emptyset\left(X_i, X_j\right) = 12\, X_i X_j^2 - X_i^4,$$

to make the test statistics scale invariant, let.

$\hat{\Delta} = \frac{\hat{\delta}_E}{\bar{X}^4}$, then (1.3)

define the symmetric kernel

$$\eta\left(X_i, X_j\right) = \frac{1}{2!}\sum_R \emptyset\left(X_i, X_j\right),$$

where the sum is over all arrangements of X_i, X_j. Then, $\hat{\Delta}$ in (13) is equivalent to the classical U-statistics, cf., Lee (1990).

$$U_n = \frac{1}{\binom{n}{3}}\sum_{i<j} \eta\left(X_i, X_j\right).$$

The following theorem summarizes the large sample properties of $\hat{\Delta}$ or U_n.

Theorem 4.2
As $n \to \infty, \sqrt{n}\left(\hat{\Delta} - \Delta\right)$ is asymptotically normal with mean 0 and variance σ^2, where
$\sigma^2 = Var\{12X\mu_2 + 12X^2\mu - X^4 - \mu_4\}$
Under H_0 the variance reduces to

$$\sigma^2 = Var\{24X + 12X^2 - X^4 - 24\} = 24768$$

Proof. The proof follows from the standard theory of U-statistics, see, e.g., Lee (1989). Using direct calculations, the mean and variance of U_n are found to be, respectively as follows:

$$\mu = E\{E(\emptyset(X_1, X_2)|X_1) + E(\emptyset(X_1, X_2)|X_2)\},$$
$$\sigma^2 = Var\{E(\emptyset(X_1, X_2)|X_1) + E(\emptyset(X_1, X_2)|X_2)\},$$

where

$$E(\emptyset(X_1, X_2)|X_1) = 12X_1\mu_2 - X_1^4,$$
$$E(\emptyset(X_1, X_2)|X_2) = 12\mu X_2^2 - \mu_4,$$

and the result follows directly.
$\eta(X) = 12X\mu_2 + 12X^2\mu - \mu_4 - X^4.$

Under H_0,
$$\eta_0(X) = 24X + 12X^2 - X^4 - 24. \quad (14)$$

From (14), it is clear that $E[\eta_0(X)] = 0$ and
$$\sigma^2 = E\left[\left(\eta_0(X)\right)^2\right] = 24768\,. \quad (15)$$

4.2 Pitman Asymptotic Efficiency (PAE)

The pitman asymptotic efficiency of the class EBUC(2) was calculated using the Linear Failure Rate (LFR), Makeham, and Weibull distributions. The Pitman efficiency is defined by

$$PAE = \frac{\mu'(\theta)}{\sigma_0} = \frac{1}{\sigma_0}\left|\frac{\partial\Delta}{\partial\theta}\right|_{\theta\to\theta_0}$$
$$= \frac{1}{\sigma_0}[12\mu\mu_2' + 12\mu' - \mu_4']$$
$$= \frac{1}{24768}[12\mu_2' + 24\mu' - \mu_4'] \quad , \quad (16)$$

where $\mu'(\theta)$ denote the partial derivative with respect to θ.
The following three families of alternatives are often used for efficiency calculation

- Linear Failure Rate: $\bar{F}(x) = \exp\left(-x - \frac{\theta x^2}{2}\right);\ \theta > 0, x \geq 0.$
- Makeham: $\bar{F}(x) = \exp(-x - \theta(x + \exp(-x) - 1));\ \theta > 0, x \geq 0.$
- Weibull: $\bar{F}(x) = \exp(-x^\theta);\ \theta > 1, x \geq 0.$

The null exponential is attained at θ= 0, 0 and 1, respectively.
The efficiency calculation for the above three alternatives are tabulated in table (1).

Table (1): Pitman Asymptotic Efficiency

Distribution	**Efficiency**
LFR	0.914
Makeham	0.238
Weibull	0.572

4.3 Critical Values

The critical value of $\hat{\Delta}$ in (13) based on 2000 simulated samples 5(1)40 are calculated. Table (2) gives the upper percentile value for 95%, 98%, 99%.

Table (2): critical value of $\hat{\Delta}$

n	95%	98%	99%
5	12.812	13.122	13.195
6	12.328	12.555	12.663
7	11.805	12.118	12.267
8	11.138	11.385	11.465
9	10.65	10.878	10.935
10	10.551	10.742	10.839
11	10.426	10.667	10.751
12	10.395	10.591	10.701
13	9.982	10.176	10.292
14	9.864	10.134	10.228
15	9.5	9.712	9.772
16	9.37	9.677	9.767
17	9.715	9.925	10.009
18	8.947	9.185	9.317
19	8.932	9.201	9.295
20	8.764	8.981	9.084
21	8.779	9.103	9.209
22	8.482	8.702	8.786
23	8.525	8.802	8.839
24	8.374	8.691	8.755
25	8.306	8.553	8.64
26	8.122	8.416	8.488
27	8.227	8.548	8.617
28	8.068	8.228	8.337
29	7.677	7.915	8.01
30	7.754	7.986	8.065
31	7.718	7.998	8.07
32	7.594	7.843	7.923
33	7.493	7.688	7.82
34	6.965	7.17	7.283
35	7.201	7.437	7.54
36	7.259	7.558	7.641
37	7.267	7.447	7.529
38	6.855	7.122	7.196
39	6.842	7.243	7.171
40	7.018	7.283	7.367

4.4 Applications

1- The following data represent 39 liver cancer's patients taken from El Minia Cancer Center Ministry of Health, Egypt, who entered the center in (1999), Abdul-Moniem (2008). The ordered life times (in days) are:
10, 14, 14, 14, 14, 14, 15, 17, 18, 20, 20, 20, 20, 20, 23, 23, 24, 26, 30, 30, 31, 40, 49, 51, 52, 60, 61, 67, 71, 74, 75, 87, 96, 105, 107, 107, 107, 116, 150.
It is found that the test statistics for the set data by using equation (13) is $\hat{\Delta}= -8.873$. This value leads to H_0 is not rejected at the significance level $\alpha = 0.05$. See Table (2). Therefore the data has not $\mathrm{EBUC}(2)$ property.

2- The following data set represents mileages for 19 military personnel that failed in service; see Abdul-Moniem (2008).
162, 200, 271, 320, 393, 508, 539, 629, 706, 778, 884, 1003, 1101, 1182, 1463, 1603, 1984, 2355, 2880.
It is found that the test statistic for the data set by using equation (13) is
$\hat{\Delta}= -7.368$. This value leads to H_0 is not rejected at the significance level$\alpha = 0.05$. See, Table (2). Therefore the data set does not have not EBUC (2) property.

References

[1] Abdul-Moniem, I. (2007). Testing EBUCA class of life distribution based on moment inequalities. The Egyptian Statist. J., 51, (2), 73 - 84.

[2] Ahmad, I. and Mugdadi, A. (2002). Further moments inequalities of life distributions with hypothesis testing applications: the IFRA, NBUC and DMRL classes. J. Statist. Planning Inf, 93, 121 - 132.

[3] Barlow, R. and Prochan, F. (1981). Statistical Theory of Reliability and Life Testing. To Begin With, Silver Spring, MD.

[4] Abouammoh, A., El-Neweihi, E., (1986). Closure of the NBUE and DMRL classes under formation of parallel system. Statist. Probab. Lett. 4, 223 – 225.
http://dx.doi.org/10.1016/0167-7152(86)90092-1

[5] Ahmad, I., Ahmed, A., Kayid, M. and El Batal, I. (2006) An ageing notion derived from the increasing convex ordering: the NBUCA class. Journal of Statistical Planning and Inference, 136, 555 - 569.
http://dx.doi.org/10.1016/j.jspi.2004.07.015

[6] Barlow, R. E., Proschan, F., (1981). Statistical Theory of Reliability and Life Testing. To Begin with, Silver Spring, MD.

[7] Cao, J. and Wang, Y. (1991). The NBUC and NWUC classes of life distributions. J. Appl. Probab., 28, 473 - 479. http://dx.doi.org/10.2307/3214882

[8] El - Batal, I. I. (2002).the EBU and EWU classes of life distribution. J. Egypt. Statist. Soc., 18 (1), 59 – 80.

[9] Hu, T. and Xie, H. (2002). Proofs of the closure property of NBUC and NBU (2) under convolution. J. Appl. Probab., 39, 224 - 227. http://dx.doi.org/10.1239/jap/1019737999

[10] Lee, A. (1989). U-Statistics. Marcell-Dekker, New York.

[11] Mahmoud, M., Attia, A. F. and Taib, I. B. (2005). On classes of exponential better than used in average .The 40en Annual conference on Stat. and Computer Science and Operations Research.

[12] Mugdadi, A. and Ahmad, I. (2005). Moment inequalities derived from comparing life with its equilibrium form. Journal of Statistical Planning and Inference, 134 (2), 303 - 317. http://dx.doi.org/10.1016/j.jspi.2004.04.016

[13] Nanda, A. Singh, H., Mitra, N. and Paul, P. (2003). Reliability properties of reversed residual life time. Communications in Statistics, Part A - Theory and Methods, 32 (10), 2031 - 2042. http://dx.doi.org/10.1081/sta-120023264

[14] Shaked, M., Shanthikumar, J. G., (2007). Stochastic Orders and their Applications, Acid-free paper, New York.

[15] Shokry, E. (2009). On Some New Aging Concepts with Applications. Ph. D. Dissertation, Cairo University.

Received: September 16, 2014; Published: November 19, 2014